2021 DSTA dys-ta.com
16th INTERNATIONAL CONFERENCE
Dynamical Systems – Theory and Applications
December 6-9, 2021, On-line

On Stability of Periodic Motion of the Swinging Atwood Machine

ALEXANDER PROKOPENYA

Warsaw University of Life Sciences – SGGW, Warsaw, Poland [ORCID 0000-0001-9760-5185]
Nowoursynowska str. 159, 02-776 Warsaw
E-mail: alexander_prokopenya@sggw.edu.pl
* Presenting Author

Abstract: We consider the swinging Atwood machine that is a conservative Hamiltonian system with two degrees of freedom. In general, it is not integrable but there exists a periodic solution of the equations of motion describing oscillations of the bodies near some equilibrium positions. An interesting peculiarity of this state of dynamic equilibrium is that owing to oscillations a body of smaller mass balances a body of larger mass. Analysing the system motion in the neighbourhood of this equilibrium, we have shown that it is stable in linear approximation. Thus, the swinging Atwood machine is an example of mechanical system the equilibrium state of which is stabilized by oscillations.

Keywords: swinging Atwood's machine, dynamic equilibrium, periodic solution, stability

1. Introduction

The swinging Atwood machine (SAM) consists of two masses $m_1, m_2 = m_1(1+\varepsilon)$ attached to opposite ends of a massless inextensible thread wound round two massless frictionless pulleys of negligible radius (see Fig. 1). The mass m_2 is constrained to move only along a vertical while mass m_1 is allowed to oscillate in a plane and it moves like a pendulum of variable length. Such a system has two degrees of freedom and its Hamiltonian function may be written in the form

$$\mathcal{H} = \frac{p_r^2}{2(2+\varepsilon)} + \frac{p_\varphi^2}{2r^2} + (1+\varepsilon)r - r\cos\varphi, \tag{1}$$

where two variables r, φ describe geometrical configuration of the system, and p_r, p_φ are the two corresponding canonically conjugate momenta.

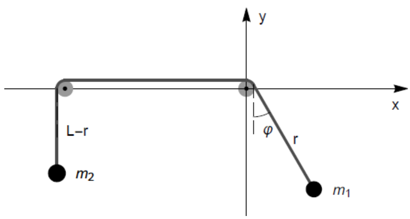

Fig. 1. The swinging Atwood machine with two small pulleys

Equations of motion of the SAM determined by the Hamiltonian (1) are essentially nonlinear, and their general solution cannot be found in symbolic form. Numerical analysis of the equations of motion has shown that, depending on the mass ratio and initial conditions, the SAM can demonstrate different types of motion (see [1, 2]). In particular, there exists a periodic solution of the equations of motion which may be represented in the form of power series in a small parameter ε (see [3])

$$r_p(t) = 1 + \frac{\varepsilon}{16}(1 + 6\cos(2t)) - \frac{\varepsilon^2}{2048}(261 + 276\cos(2t) + 105\cos(4t)) + \cdots, \quad (2)$$

$$\varphi_p(t) = \sqrt{\varepsilon}\left(2\cos t - \frac{53\varepsilon}{192}\cos(3t) + \frac{\varepsilon^2}{16384}\left(2959\cos t + 1699\cos(3t) + \frac{5813}{5}\cos(5t)\right) + \cdots\right). \quad (3)$$

The existence of periodic solution (2)-(3) means that for given value of parameter ε one can choose such initial conditions that the system is in the state of dynamical equilibrium when the bodies oscillate near some equilibrium positions. Note that for $\varepsilon > 0$ the system under consideration has no a state of static equilibrium when the coordinates $r(t), \varphi(t)$ are some constants. The main purpose of this talk is to investigate whether the system will remain in the neighbourhood of the equilibrium if the initial conditions are perturbed or whether the periodic solution (2)-(3) is stable.

2. Stability Analysis

To investigate stability of periodic solution (2)-(3) we introduce small perturbations q_1, p_1, q_2, p_2 of the solution and expand the Hamiltonian (1) into power series in terms of q_1, p_1, q_2, p_2 up to the second order inclusive. Then equations of the perturbed motion may be written in linear approximation in the canonical form with the Hamiltonian

$$\mathcal{H}_2 = \frac{p_1^2}{2(2+\varepsilon)} + \frac{p_2^2}{2r_p^2} + \frac{p_{\varphi 0}^2}{2r_p^4}q_1^2 + \frac{r_p}{2}\cos\varphi_p\, q_2^2 - \frac{2p_{\varphi 0}}{r_p^3}q_1 p_2 + q_1 q_2 \sin\varphi_p,$$

where $p_{\varphi 0}$ is the momentum canonically conjugate to solution φ_p. It is clear that the perturbed motion of the system is determined by the linear system of four differential equations with periodic coefficients, and their general properties have been studied quite well (see [4]). The behavior of solutions to such equations is determined by its characteristic exponents which may be found in the form of power series in ε. Doing necessary symbolic computations, we have found two pairs of purely imaginary characteristic exponents up to the second order in ε

$$\lambda_{1,2} = \pm i, \quad \lambda_{3,4} = \pm i\frac{\sqrt{3\varepsilon}}{2}\left(1 - \frac{17\varepsilon}{32} + \frac{85}{256}\varepsilon^2\right).$$

According to Floquet-Lyapunov theory (see [4]), the corresponding solutions to differential equations with periodic coefficients describe the perturbed motion of the system in the bounded domain in the neighbourhood of the periodic solution (2)-(3). It means that this solution is stable in linear approximation, and so the SAM is an example of mechanical system the equilibrium state of which is stabilized by oscillations.

References

[1] TUFILLARO N.B., ABBOTT T.A., GRIFFITHS D.J.: Swinging Atwood's machine. *American Journal of Physics* 1984, **52**(10):895-903.

[2] PROKOPENYA A.N.: Motion of a swinging Atwood's machine: simulation and analysis with Mathematica. *Mathematics in Computer Science* 2017, **11**(3):417-425.

[3] PROKOPENYA A.N.: Construction of a periodic solution to the equations of motion of a generalized Atwood's machine using computer algebra. *Programming and Computer Software* 2020, **46**(2):120-125.

[4] YAKUBOVICH V.A., STARZHINSKII V.M.: *Linear Differential Equations with Periodic Coefficients.* John Wiley: New York, 1975.