2021 DSTA dys-ta.com
16th INTERNATIONAL CONFERENCE
Dynamical Systems – Theory and Applications
December 6-9, 2021, On-line

Brachistochrone Problem with State Constraints of a Certain Type,

CHERKASOV OLEG[1], SMIRNOVA NINA[2*]

1. Lomonosov Moscow State University, Moscow, Russia [0000-0003-1435-1541]
2. Lomonosov Moscow State University, Moscow, Russia [0000-0002-6532-8037]
* Presenting Author

Abstract: The problem of maximizing the horizontal coordinate of a point mass moving in a vertical plane under the action of gravity forces, viscous friction, support reaction of the curve and thrust is considered, as well as the interrelated Brachistochrone problem. Two cases are addressed. The first is when the thrust applied is constant. The second is when the penalty for the control expenditures is included in the goal function. Assumed that inequality-type constraints are imposed on the slope angle of the trajectory. The system of equation belongs to a certain type that allows reduce the optimal control problem with state constraints to the optimal problem with control constraints. The maximum Principle procedure is applied, and the qualitative analysis of the boundary-value problem is presented. As a result, the sequence and the number of the arcs with motion along the phase constraints are determined and the synthesis of the optimal control is designed. The results of numerical simulation for the case of quadratic resistance are presented to illustrate the theoretical conclusions. It is shown that optimal trajectory of the Brachistochrone problem with viscous friction contains no more than one section of motion along the lower constraint and no more than two sections of motion along the upper one. For the frictionless Brachistochrone the extremal trajectory reaches for each constraint no more than once.

Keywords: brachistochrone problem, state constraints, viscous friction, qualitative analysis

1. Introduction

The presence of state constraints significantly complicates the study of optimal control problems. An effective solution can be designed if the structure of the optimal trajectory, the number of arcs with motion along the constraints and their sequence are determined. In this paper, the approach that allows one to construct an optimal synthesis for state constraints of a certain type is demonstrated using Brachistochrone problems with thrust.

Consider the motion of a material point in a vertical plane in a homogeneous field of gravity forces and in a homogeneous resisting medium. The problem is to determine the shape of the trajectory that maximizes the horizontal coordinate of a point when it is transferred from a given initial state to a given height in a fixed time interval. Along with the problem of maximizing the range, the minimum-time problem is considered: the problem of choosing the shape of the trajectory connecting two given points of the vertical plane, the travel time along which will be minimal.

The minimum-time problem in the considered formulation is called the Brachistochrone problem [1]. The frictionless brachistochrone with state constraints in the form of a linear function imposed on the coordinates of a point was considered in [2-3]. In [2], an analytical solution is proposed under the assumption that the solution contains single arc with motion along the state constraint. In [3] the Brachistochrone problem serves as an example of the efficiency of numerical methods for solving problems with state constraints. In [4], the problem of optimal maneuver over the lunar surface is reduced to the Brachistochrone problem with a constraint on the trajectory inclination angle.

The purpose of this article is the qualitative analysis of the range maximization problem in a given time in the presence of viscous resistance, thrust and state constraints on the trajectory inclination angle. As a result of this analysis, it is possible to construct an optimal synthesis, to determine the number and sequence of the arcs with motion along the state constraints.

2. Problem Formulation

Equations of motion in dimensional variables are as follows:

$$\begin{cases} \dot{x} = v\cos\theta, \\ \dot{y} = v\sin\theta, \\ \dot{v} = -v - \sin\theta + p. \end{cases} \tag{1}$$

Here x, y are horizontal distance and vertical altitude, respectively, v is the module of the velocity, p is the thrust, subjected to inequality $-\overline{p} \le p \le \overline{p}$, where $\overline{p}$ is a positive constant, θ is the slope angle. θ and p are considered as control variables.

Boundary conditions have the form:

$$x(0) = x_0, y(0) = y_0, v(0) = v_0, y(T) = y_T \tag{2}$$

where T is final time (is considered as given). The cost function has the following form

$$J = -x(T) + \int_0^T p^2 dt \to \min \tag{3}$$

State constraints are as follows:

$$\theta(t) \in [\theta_1, \theta_2] \tag{4}$$

where θ_1, θ_2 are constants. The problem (1) - (4) is Mayer optimal control problem.

References

[1] HERMAN H. GOLDSTINE: *A history of the calculus of variations from the 17 th through the 19 th century, Studies in the History of Mathematics and Physical Sciences, Vol. 5,* Springer-Verlag, New York-Heidelberg-Berlin, 1980.

[2] A.E. BRYSON, Y.C.HO: *Applied Optimal Control,* Blaisdell Publishing Company, Waltham, Massachusetts, 1969..

[3] B. C. FABIEN: Numerical Solution of Constrained Optimal Control Problems with Parameters, *Applied Math. and Computation* 1996,80:43-62.

[4] R.K. CHENG, D. A. CONRAD: Optimum translation and the brachistochrone, *American Institute of Aeronautics and Astronautics (AIAA), Aerospace Sciences Meeting,* 1964, 49 DOI:10.2514/6.1964-49.