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# Brachistochrone Problem with Variable Mass,

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**Abstract:** The motion of a point mass in a vertical plane under the action of gravity forces, viscous friction, support reaction of the curve and the thrust is considered. The slope angle and the thrust are treated as control variables. The amount of the propellant is given. The aim is to maximize the horizontal coordinate of the particle. Time of the process is given. The interrelated Brachistochrone problem is also considered. For the case of frictionless motion, it is shown that optimal thrust control is bang-bang-type, and trajectory consists of two arcs, starting with maximum thrust, and ending with zero thrust. Optimal synthesis in the three dimensional space “mass-velocity-slope angle” is designed. For the case of linear viscous friction the arc with singular thrust includes in the extremal trajectory. It is shown that optimal thrust program consists of either two arcs, maximum thrust at the beginning and zero thrust at the end, or three arcs: maximum thrust at the beginning, then intermediate (singular) thrust and zero thrust at the end. The control logic of the thrust is similar to the Goddard problem. The results of numerical simulation for the case of linear viscous friction illustrating the theoretical conclusions are presented.

**Keywords:** singular arc, thrust control, brachistochrone problem, viscous friction

## 1. Introduction

The motion of a material point in a vertical plane in a homogeneous field of gravity and in a homogeneous, resisting medium is considered. The trajectory angle and thrust are considered as control variables. The goal of the control is to maximize the horizontal range for a given time. The amount of fuel is given. Along with the range maximization problem, we can consider a modified brachistochrone problem formulated as follows: find a curve connecting two points in the vertical plane along which a material point in the field of gravity and nonconservative force moves from the initial to the final point in the shortest time.

The classical theory of the calculus of variations and, later, the theory of optimal control were applied to the problem of maximizing the vertical altitude of a rocket with a given amount of fuel. Two special cases, namely, one with a linear dependence of the resistance on the velocity, and the other with a quadratic dependence on the velocity, were considered in [1]. In [2], the optimal flight in the vertical plane for an "intermediate" model of an aircraft was studied. The slope angle was taken as control variable. This model is suitable for studying the optimal motion of special types of aircraft, for which it is possible to change the lifting force without changing the drag force. For various modifications of the brachistochrone problem with viscous friction, the normal component of the reaction force of the curve also allows changing the angle of inclination of the trajectory without changing the resistance force [3]. The Results of numerical simulation for the case of an accelerating force proportional to the velocity were presented in [3]. The Brachistochrone problem in the presence of a constant thrust force and a linear viscous friction force was studied in [4]. In [5], the problem of maximizing the horizontal range with a penalty on fuel consumption was considered, while it was assumed that the change in the amount of fuel does not affect the dynamics of the point movement.

In this paper, the problem of maximizing the range is considered taking into account the influence of the amount of fuel on the dynamics of the point mass.

## 2. Problem Formulation

Equations of motion are as follows:

$$\begin{cases} \dot{x} = v\cos\theta, \\ \dot{y} = v\sin\theta, \\ \dot{v} = \dfrac{-kv + cu}{m} - g\sin\theta, \\ \dot{m} = -u; \end{cases} \tag{1}$$

where $x, y$ are horizontal and vertical coordinates of the particle correspondingly, $v$ is the velocity modulus, $m$ is mass of the particle, $k$ is a coefficient of the viscous friction, $c$ is exhaust velocity of the gas flow, $g$ is gravitational acceleration, $\theta$ is the slope angle, $u$ is mass change rate, $\theta$ and $u$ are considered as control variables. Boundary conditions for the system (1) have the form:

$$x(0) = x_0, y(0) = y_0, v(0) = v_0, m(0) = m_0, m(T) = m_T. \tag{2}$$

$T$ is given process time. The goal function is

$$J = -x(T) \to \min_{\theta,u} \tag{3}$$

$u \in [0, \bar{u}]$, $\bar{u}$ is a positive constant. The problem (1) - (3) is Mayer optimal control problem.

## References


[1] TSIEN, H. S., EVANS, R. C: Optimum Thrust Programming for a Sounding Rocket, *Journal of American Rocket Society Journal* 1951, 21,(5):99-107.

[2] MENON, P.K.A., KELLEY, H.J., AND CLIFF, E.M: Optimal Symmetric Flight with an Intermediate Vehicle Model, *J. Guidance* c1985, 8,(3):312-319.

[3] VRATANAR, B., SAJE, M: On the Analytical Solution of the Brachistochrone Problem in a Non-conservative Field, *Int. J. Non-Linear Mechanics* 1998, 33(3):489-505.

[4] DRUMMOND, J.E., DOWNES, G.L: The Brachistochrone with Acceleration: A Running Track, *Journal of Optimization Theory and Applications* 1971, 7(6): 444-449.

[5] CHERKASOV, O., ZARODNYUK, A., SMIRNOVA N: Optimal Thrust Programming along the Brachistochronic Trajectory with Non-Linear Drag, *International Journal of Nonlinear Sciences and Numerical Simulation, Freund Publishing House Ltd.(Israel)*,2019, 20(1):1-6.