Quantum spectrum of Isochronous oscillatory motions and the WKB method

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1 Introduction

Necessary and sufficient conditions for isochrony of oscillatory motions introduced in Physica Scripta vol 94, N 12 are discussed. Thanks to the WKB perturbation method expressions are derived for the corrections to the equally spaced valid for analytic isochronous potentials. In this paper, we bring some improvements of these results and we suggest another quantization of the quantum spectrum. This result will be illustrated by several examples

It is in general not trivial to solve in the case of complex potentials, apart from numerical resolution. There is however an approximate method of resolution, the WKB approximation, named after the physicists Wentzel, Kramers and Brillouin. This approximation is based on the fact that the solutions of the Schrodinger equation can be approximated by a function comprising usually conventional quantities, provided that the potential does not vary strongly over distances of the order of the length of wave.

These is a link between classical and quantum transformations. This fact has been established by Eleonskii and al. [2]. They show that the classical limit of the isospectral transformation for the Schrodinger equation is precisely the isochronicity preserving the energy dependence of the oscillation frequency. In quantum mechanics, the energy levels of a parabolic well are regularly spaced by a certain quantity. Moreover, it is possible to construct potentials, essentially different from the parabolic well, whose spectrum is exactly harmonic.
Our method described below permits also another approach of two-dimensional superintegrability.

Consider the scalar equation with a center at the origin 0

$$\ddot{x} + g(x) = 0 \tag{1}$$

or its planar equivalent system

$$\dot{x} = y, \qquad \dot{y} = -g(x) \tag{2}$$

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where $\dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}$ and $g(x) = \frac{dG(x)}{dx}$ is analytic on R where $G(x)$ is the potential of (1).

Suppose system (2) admits a periodic orbit in the phase plane with energy E and $g(x)$ has bounded period for real energies E. Given $G(x)$, Let $T(E)$ denotes the minimal period of this periodic orbit. Its expression is

$$T(E) = 2\int_a^b \frac{dx}{\sqrt{2E - 2G(x)}}. \qquad (3)$$

$T(E)$ is well defined and there is a neighborhood of the real axis for which $T(E)$ is analytic.

We suppose that the potential $G(x)$ has one minimum value which, for convenience locate at the origin 0 and $\frac{d^2G(x)}{dx^2}(0) = 1$. The turning points a, b of this orbit are solutions of $G(x) = E$.
Then the origin 0 is a center of (2). This center is isochronous when the period of all orbits near $0 \in R^2$ are constant ($T = \frac{2\pi}{\sqrt{g'(0)}} = 2\pi$). The corresponding potential $G(x)$ is also called isochronous.

Since the potential $G(x)$ has a local minimum at 0, then we may consider an involution A by

$$G(A(x)) = G(x) \ and \ A(x)x < 0$$

for all $x \in [a, b]$. So, any closed orbit is A-invariant and A exchanges the turning points: $b = A(a)$.

2 An alternative result

As consequences we prove the following

Theorem 2-1 *Let $G(x) = \int_0^x g(s)ds$ be an analytic potential defined in a neighborhood of 0. Suppose equation*

$$\ddot{x} + g(x) = 0 \qquad (1)$$

has an isochronous center at 0. Let $g^{(n)}(x)$ be the n-th derivative of the potential (with respect to x): $g^{(n)}(x) = \frac{d^n}{dx^n}G(x)$, $n \geq 1$ then these derivatives may be expressed under the form

$$g^{(n)}(x) = a_n(G)x + b_n(G), \quad n \geq 0 \qquad (4)$$

where a_n and b_n are analytic functions with respect to G.