Global sliding mode control design for a 3D pendulum

João Francisco Silva Trentin [1], Davi Antônio dos Santos [1]

[1]Division of Mechanical Engineering, Aeronautics Institute of Technology (ITA), São José dos Campos, SP, Brazil. [ORCID: 0000-0002-6726-8699, 0000-0001-5995-3103]

Abstract: The dynamics and control of pendulums attract attention of researchers for a long time. Their mathematical models can approximate many real-world applications besides being very helpful for pedagogical reasons. This paper studies the attitude control of a 3D pendulum using a torque control law based on an unit-vector multi-input global sliding mode control. Numerical simulations are carried out to evaluate the proposed controller and the results are compared to a proportional-derivative feedback linearization control law.

Keywords: nonlinear dynamics, 3D pendulum, global sliding mode control.

1. Introduction

Inverted pendulum-like systems have been deeply studied and until today provide a rich source of nonlinear dynamical systems. The 3D pendulum is a rigid body fixed to a pivot, allows three rotational degrees of freedom and is fully actuated by three control torques [1]. This dynamical system has been used as a benchmark in the past decades for investigating new control techniques, but there are still some open points with respect to robust control. Sliding mode control (SMC) techniques can deal with multi-variable dynamics, nonlinearities, and actuator constraints [2]. It has attractive properties such as insensitivity to bounded uncertainties, disturbances, as well as parasitic dynamics [2]. The present work is concerned with the design of a global sliding mode control (GSMC) law for the 3D pendulum. The GSMC can also be referred to as integral sliding mode control. In this case, the technique designs a sliding manifold that guarantees robustness to bounded uncertainties from the initial condition and there is no reaching phase in such design. Thus, this paper presents briefly the mathematical model of the 3D pendulum, the adopted sliding surface as well as the proposed control law. Numerical results demonstrate the effectiveness of the method.

2. Methodology and Results

The modified Rodrigues parameters (MRPs) are used to describe the attitude of the 3D pendulum to avoid singularities. The MRP $\mathbf{p} \in \mathbb{R}^3$ is defined as $\mathbf{p} \triangleq \tan\frac{\phi}{4}\hat{\mathbf{e}}$, where $\phi \in (-2\pi, 2\pi)$ is the principal angle rotation, $\hat{\mathbf{e}}$ is the principal axis unit vector referring to Euler's principal rotation theorem. The kinematic differential equation of the MRPs is:

$$\dot{\mathbf{p}} = \frac{1}{4}\mathbf{\Sigma}(\mathbf{p})\boldsymbol{\omega}, \tag{1}$$

where $\mathbf{\Sigma}(\mathbf{p}) \in \mathbb{R}^{3\times 3}$ given by $\mathbf{\Sigma}(\mathbf{p}) = (1 - \mathbf{p}^T\mathbf{p})\mathbf{I}_3 + 2[\mathbf{p}\times] + 2\mathbf{p}\mathbf{p}^T$ and $\boldsymbol{\omega}$ is the body angular velocity. Based on the angular momentum of the 3D pendulum, it is possible to obtain the following equation of motion:

$$\mathbf{J}\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \mathbf{t}_g + \mathbf{u} + \mathbf{d}, \tag{2}$$

where $\mathbf{J} \in \mathbb{R}^{3\times 3}$ is the inertia matrix of the 3D pendulum, $\dot{\boldsymbol{\omega}} \in \mathbb{R}^3$ is the body angular acceleration, $\mathbf{u} \in \mathbb{R}^3$ the control torque vector, and $\mathbf{d} \in \mathbb{R}^3$ represents the unknown disturbances, where $\|\mathbf{d}\| \leq \rho$, with known $\rho \in \mathbb{R}_+$. $\mathbf{t}_g = mg\mathbf{r}_{cm} \times \mathbf{D}\hat{n}_3$ provides the external torque caused by gravity, where m is the mass, $\mathbf{r}_{cm} \in \mathbb{R}^3$ the center of mass vector, $\mathbf{D}$ the attitude matrix, and $\hat{n}_3$ the direction that gravity acts. The complete model of the 3D pendulum can be rewritten in terms of the attitude and dynamics errors

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}), \tag{3}$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}) + \mathbf{B}(\mathbf{u} + \mathbf{d}), \tag{4}$$

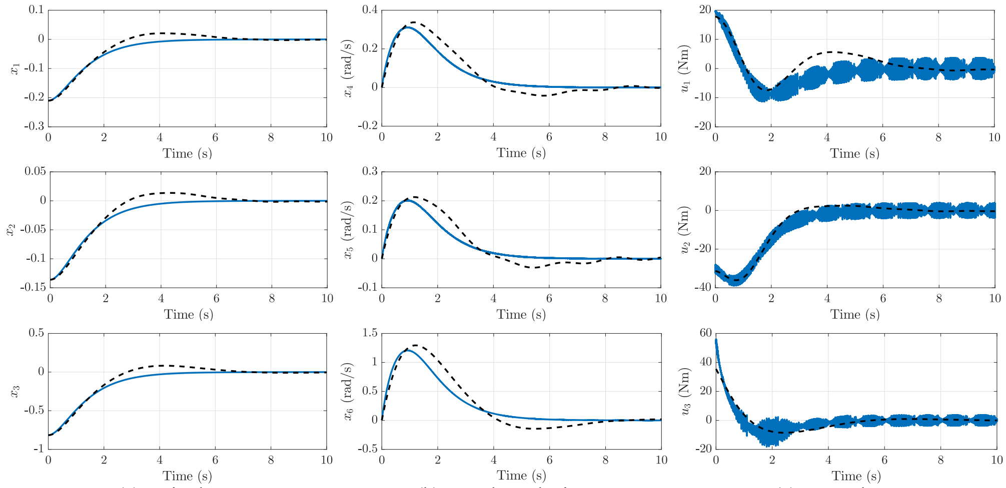
where $\mathbf{x} \triangleq (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^6$ is the state errors vector, $\mathbf{x}_1 \in \mathbb{R}^3$ denotes the attitude error in terms of the MRP vector, and $\mathbf{x}_2 \in \mathbb{R}^3$ is the angular velocity error. $\mathbf{f}_1(\mathbf{x})$, $\mathbf{f}_2(\mathbf{x})$, and $\mathbf{B}$ are continuous vector and matrix fields such that $\mathbf{f}_1 : \mathbb{R}^6 \to \mathbb{R}^3$, $\mathbf{f}_2 : \mathbb{R}^6 \to \mathbb{R}^3$, $\mathbf{B} : \mathbb{R}^6 \to \mathbb{R}^{3\times 3}$, and $\|(\partial\mathbf{f}_1/\partial\mathbf{x}_2)\mathbf{B}\| \neq 0, \forall\mathbf{x} \in \mathbb{R}^6$. The control objective is to design a control law $\mathbf{u}$ which makes $\mathbf{x} = 0$ a global exponential stable equilibrium point of (3) and (4). To do so, the following time-varying sliding function can be defined

$$\mathbf{s}(t, \mathbf{x}) \triangleq \boldsymbol{\sigma}(t) - \mathbf{P}(t)\boldsymbol{\sigma}(0), \tag{5}$$

where $\boldsymbol{\sigma}(t) \triangleq \mathbf{C}\mathbf{x}_1 + \mathbf{f}_1(\mathbf{x}) \in \mathbb{R}^3$, with $\mathbf{C} \in \mathbb{R}^{3\times3}$ being a given diagonal matrix, and $\mathbf{P} : \mathbb{R}_+ \to \mathbb{R}^{3\times3}$ is a function which satisfies, among other design conditions, $\mathbf{P}(0) = \mathbf{I}_3$ [3]. Considering the set $\mathscr{S} \triangleq \{\mathbf{x} \in \mathbb{R}^6 : \mathbf{s}(t, \mathbf{x}) = 0, \forall t \geq 0\}$ the eventual sliding set and to ensure a global sliding mode of (3) and (4) in $\mathscr{S}$, the following control law is designed [3]

$$\mathbf{u} = -\left(\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2}\mathbf{B}\right)^{-1}\left(\left(\mathbf{C} + \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1}\right)\mathbf{f}_1 + \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2}\mathbf{f}_2 + \dot{\mathbf{P}}(t)\boldsymbol{\sigma}(0) + \kappa\frac{\mathbf{s}}{\|\mathbf{s}\|}\right), \tag{6}$$

where $\kappa > \|(\partial \mathbf{f}_1/\partial \mathbf{x}_2)\mathbf{B}\|\rho$ is a design parameter. Figure 1 shows the numerical results for the attitude control of the 3D pendulum presenting the attitude and angular velocity errors as well as the control torques. The goal is to drive the 3D pendulum to the upright (inverted) position and stabilise it there. We can observe that the errors for the GSMC law go to zero faster than the PD control law. Disturbances are applied as sinusoidal signals with frequency of 0.5 Hz and amplitude 0.1 Nm. The parameters used for the GSMC are $\kappa = 0.18$, $\mathbf{C} = \mathbf{I}_3$, and $\mathbf{P}(t) = \exp(-2t)\mathbf{I}_3$. For the PD control law, $\mathbf{K}_1 = 3\mathbf{I}_3$ is the proportional gain and $\mathbf{K}_2 = \mathbf{I}_3$ the derivative gain.



(a) Attitude error (b) Angular velocity error (c) Control torques

Figure 1: Numerical results for the attitude control of the 3D pendulum in the upright position, where - - - PD control and — GSMC law.

3. Final remarks

This work has presented the design of a global sliding mode control law for the 3D pendulum and compared the results to a PD control law. The GSCM exhibited a better performance taking the 3D pendulum to the upright position faster and presenting no overshoot.

Acknowledgements: The authors thank the São Paulo Research Foundation (FAPESP) for the financial support (grants 2019/05334-0 and 2020/12314-3). The third author is also grateful for the support of CNPq/Brazil (grant 302637/2018-4).

References

1. Qijia Yao, 'Robust adaptive finite-time attitude tracking control of a 3d pendulum with external disturbance: Numerical simulations and hardware experiments,' Nonlinear Dynamics, Vol 102, No 1, pp 223–239, 2020.
2. Yuri Shtessel, Christopher Edwards, Leonid Fridman and Arie Levant, *Sliding mode control and observation*. Springer, 2014, vol. 10.
3. A. Bartoszewicz, 'Time-varying sliding modes for second-order systems,' IEE Proceedings - Control Theory and Applications, Vol 143, pp 455–462(7), 5 Sep. 1996.