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Generalized Neimark—Sacker Bifurcations

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Abstract: We sketch a proof of a new type of bifurcation for discrete dynamical systems in all finite dimensions that generalizes the Neimark—Sacker bifurcation. In addition, we identify an application to discrete predator-prey dynamics.

Keywords: sink, source, bifurcation parameter, homeomorph

1. Introduction

The Neimark—Sacker bifurcation is a rather ubiquitous codimension-1 feature for smooth discrete dynamical systems on smooth surfaces (see [2] – [5]). It can be essentially described as follows: A fixed point spiral sink on a smooth surface changes to a fixed spiral source at the bifurcation value of a single real parameter λ and generates an invariant smooth diffeomorph of the unit circle $\mathbb{S}^1$, which encloses the fixed point and expands with increasing λ. We introduced a generalization in [1], which has the following characterization: If a sink fixed point of a smooth discrete dynamical system on a smooth n-dimensional manifold M changes to a source at the bifurcation value of a single real parameter λ, it generates an invariant homeomorph of the unit (n-1)-sphere $\mathbb{S}^{n-1}$, which encloses the fixed point and expands with increasing λ.

2. Results and Discussion

Our main result is the following:

Theorem

Suppose that

$$f: U \times (-a,a) \to \mathbb{R}^n, (x,\lambda) \to f(x,\lambda) := f_\lambda(x), \qquad (1)$$

where U is an open set in $\mathbb{R}^n$ containing the origin and $a > 0$, is a smooth map depending on the real parameter λ such that: (i) $f(0,\lambda) = 0$ *for all* $\lambda \in (-a,a)$; (ii) *all eigenvalues of the derivative* $f'_\lambda(0)$ *are interior or exterior to the unit circle in the complex plane when* $\lambda < 0$ *or* $\lambda > 0$ (*so that* $\lambda = 0$ *is a bifurcation value*), *respectively*; *and* (iii) $f\left(U \times (-a,a)\right) \subset U$.

Then, for λ > 0 *there is an f-invariant homeomorph* S_λ *of* $\mathbb{S}^{n-1}$ *enclosing the origin with a diameter that increases with* λ.

Proof Sketch. For each small positive value of λ, start with a correspondingly sized (*n*-1)-sphere $S_0(\lambda)$. It can be shown, with some difficulty, that

$$f_\lambda^m\left(S_0(\lambda)\right) \to S_\lambda$$

as $m \to \infty$ which is homeomorphic to the (*n*-1)-sphere and *f* – invariant. □

As shown in [1], this theorem has significant applications to discrete dynamical systems models for population dynamics, and is likely be useful for analyzing several other phenomena.

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3. Concluding Remarks

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References

[1] JOSHI Y, SAVESCU M, SYED M, BLACKMORE D: Interesting features of three-dimensional discrete Lotka—Volterra dynamics..(*preprint*)

[2] KUZNETSOV Y.A.: *Elements of Bifurcation Theory*. Springer-Verlag: New York, 1994.

[3] NEIMARK J: On some cases of periodic motion depending on parameters. *Dokl. Akad Nauk SSSR* **129** (1959), 36-39.

[4] SACKER R: On invariant surfaces and bifurcations of periodic solutions of ordinary differential equations. *Rep. IMM-NYU* **333** (1964), 1-62.

[5] WIGGINS S.: *Introduction to Applied Nonlinear Dynamical Systems and Chaos, 2nd ed.*. Springer-Verlag: New York, 2003.