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Nonlinear Oscillations of an Elastica Between Cylindrical Boundaries

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Abstract: In this work we consider the free and forced dynamics of a fixed-free elastica in contact with rigid cylindrical boundary. In addition to the geometric nonlinearity due to deformation, the boundaries introduce nonlinearity due to change in effective beam length as it deforms. We consider both unilateral and bilateral constraints and explore the realization of nonlinear normal modes (NNMs) and their forced dynamics in this class of dynamical systems. The current study invokes Galerkin's method, multi scale analysis and harmonic balancing.

Keywords: Elastica, nonlinear oscillations, Galerkin's method, perturbation analysis

1. Introduction

It is well known that the time period of oscillations of a simple pendulum is amplitude dependent. Owing to Huygen's (1656) ingenuity of varying the effective length of string as it wraps/unwraps around (cycloidal) boundary, the oscillations were rendered isochronous. In the late twentieth century, researchers started investigating mechanical elements like flexures as they wrap/unwrap around obstacles to exhibit nonlinear dynamical behaviour. Fung et al. [1] considered Euler-Bernoulli beam with rigid cylindrical boundary on one side and investigated the nonlinearity induced considering no loss of energy during contact. Crespo da Silva et al. [2,3] developed a nonlinear beam model and explored the free and forced nonlinear oscillations and the resonances thereof.

In the current study we consider a nonlinear beam [2,3] and introduce smooth cylindrical boundaries unilaterally and bilaterally to study their effect on its nonlinear dynamics.

2. Mathematical Modelling

Consider a flexure (large slenderness ratio) with the assumptions, (i) inextensible: $(1+u_s)^2+v_s^2=1$, (ii) no warping/shear deformation: $v_s(1+u_s)^{-1}=\tan(\theta)$. The planar motion is described by two displacements $u(s,t)$ and $v(s,t)$ as shown in Fig. 1. 'A' is the point of loss of contact of the beam with the cylindrical boundary and s at A is $\gamma(t)$. The Lagrange multiplier Λ incorporates the geometric constraint C in domain $0 \le s \le \gamma^-$ and since γ^- is varying, the Hamilton's principle in the general noncontemporaneous form will be used in the following, (where $v_s = \partial v/\partial s$, …)

$$\delta\int_{t_1}^{t_2}\left\{\left(\left(\int_0^{\gamma^-}+\int_{\gamma^+}^{l}\right)\frac{1}{2}\{\rho(u_t^2+v_t^2)-\beta\theta_s^2+\lambda\big(1-(1+u_s)^2-v_s^2\big)\}ds+\left(\int_0^{\gamma^-}\Lambda C\,ds\right)\right\}dt=0 \tag{1a}$$

$$C(v,s)=s^2+v^2-2a|v|=0 \text{ and } |v|=a-\sqrt{a^2-s^2};\ 0\le s\le\gamma^- \tag{1b}$$

$$v(0,t)=v_s(0,t)=v_{ss}(L,t)=v_{sss}(L,t)=0 \tag{1c}$$

In the absence of the cylindrical boundary, i.e., $\Lambda=0$, the nondimensional equation of motion considering cubic nonlinear terms is given by

$$w_{\tau\tau}+w_{zzzz}=-(w_z(w_zw_{zz})_z)_z-\frac{1}{2}\left(w_z\int_1^z\int_0^z(w_z^2)_{\tau\tau}\,dz\,dz\right)_z \tag{2}$$

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Where $w = v/l, z = s/l, \tau = t\sqrt{\beta/\rho l^4}$ and Lagrange multiplier, $\lambda = -w_{zzz}w_z - \int_1^z \int_0^z (w_z^2)_{\tau\tau}\, dz\, dz$. Analysis of (2) with BCs (1c) is discussed later. For $\Lambda \neq 0$, additional conditions are imposed:

a) *Perpendicularity*: A vector $\vec{R} = s\hat{\imath} + (a - |v|)\hat{\jmath}$ from the centre of the cylindrical boundary to a point s on the neutral axis is perpendicular to the tangent at s, i.e., $\vec{R} \cdot \vec{R}_s = 0;\, 0 \leq s \leq \gamma^-$

b) *Continuity*: The displacement and the slope are continuous at $s = \gamma$

c) *Velocity at* $s = \gamma$: The total time derivative of $v(\gamma(t), t)$ is zero implying, $v_t(\gamma, t) = 0$

3. Results and Discussions

For $\Lambda = 0$, considering Galerkin's method, we choose $\phi_{(i)}(z)$, the i^{th} (mass) normalized eigenfunction corresponding to a linear beam, as a comparison function. Letting $w(z,\tau) = \phi_{(i)}(z)\,\eta_{(i)}(\tau)$, the corresponding nonlinear oscillator (disregarding the modal interactions at this stage) is,

$$\ddot{\eta}_{(i)} + \omega^2\eta_{(i)} + G_1\eta_{(i)}^3 + G_2\eta_{(i)}\left(\eta_{(i)}\ddot{\eta}_{(i)} + \dot{\eta}_{(i)}^2\right) = 0$$

$$G_1 = \int_0^1 \phi_{(i)}\left(\phi'_{(i)}\left(\phi'_{(i)}\phi''_{(i)}\right)'\right)' dz\,;\; G_2 = \int_0^1 \phi_{(i)}\left(\phi'_{(i)}\int_1^z\int_0^z \phi'^2_{(i)}\, dz\, dz\right)' dz \tag{3}$$

The frequency-amplitude relation (Fig. 2) is derived by harmonic balancing [4]. Interestingly, the first mode shows near isochronous oscillations, whereas the higher modes exhibit amplitude dependence.

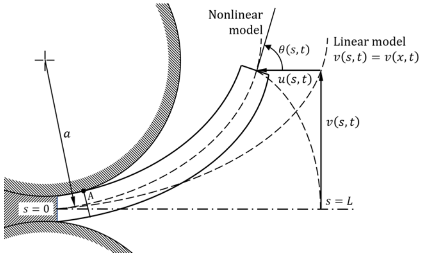

Fig. 1. Kinematics of the nonlinear beam (***a*** is the sum of cylinder radius and half beam thickness)

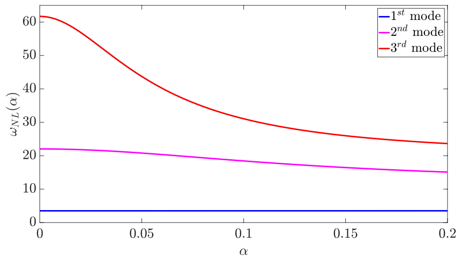

Fig. 2. Amplitude-frequency relation

4. Concluding Remarks

This study considers nonlinear oscillations of a thin elastica oscillating between two rigid cylinders. The amplitude dependent frequency is shown here for few modes in the absence of boundaries. A complete analysis with unilateral and bilateral cylindrical boundaries will be presented in the full version. Additionally, we discuss about the NNMs, their existence and stability in this class of structures. Forced dynamics is studied for a simple point load and the excitation NNMs.

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