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# Application of multiple scales method to the problem of plane pendulum motion with extended damping model

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**Abstract:** In the paper, the plane motion of a physical pendulum involving the interactions with the surrounding air is considered. These interactions are described employing the model consisting of three components. The linear and quadratic terms are proportional to the magnitude of the velocity and its square, respectively. The last component is proportional to the tangential component of the acceleration. According to the semi-empirical Morison equation, the quadratic term and acceleration dependent component depict the total force exerted on the body i.e. the drag force and inertia force including the concept of mass added. The multiple scales method (MSM) is used to obtain the approximate asymptotic solution to the problem. A slight change in the natural frequency is caused by the inertial component of the total damping force. In turn, the occurrence of the absolute value of velocity in the damping model complicates the solving procedure. The accuracy of solutions obtained using the multiple scales method is compared with the experimental results.

**Keywords:** damping model, physical pendulum, method of multiple scales

## 1. Mathematical Model

The plane motion of the physical pendulum of mass *m* and length *L* is investigated. The damping force can be presented as the power series of the velocity magnitude [1]:

$$\vec{F}(v) = -(c_0 + c_1 v + c_2 v^2 + \cdots)\frac{\vec{v}}{v}, \tag{1}$$

where $\{c_i\}_{i=0}^{\infty}$ are constants. The concept of extending the model with a term depending on the tangential component of the acceleration was inspired by the Morison equation used in hydromechanics. Taking into account only the second and third terms of series (1) and the component related to the added mass, one can obtain the following dimensionless equations of motion

$$\ddot{\varphi}(\tau) + \sin\varphi(\tau) + \alpha_1\dot{\varphi}(\tau) + \alpha_2\dot{\varphi}(\tau)|\dot{\varphi}(\tau)| = 0 \tag{2}$$

where

$\tau = \hat{\omega}\, t,\quad \hat{\omega} = \sqrt{\frac{mgL}{6I_0+4c_aL^2}},\quad I_0$ – the mass moment of inertia, $c_a$ – the inertia coefficient, $\alpha_1, \alpha_2$ – the dimensionless damping coeficients,

The following initial conditions with known quantities $\varphi_0, \omega_0$ supplement Eq. (2)

$$\varphi(0) = \varphi_0, \qquad \dot{\varphi}(0) = \omega_0. \tag{3}$$

## 2. Asymtotic Solution

The approximate analytical solution to the initial value problem (2)–(3) is obtained using MSM [2]. The system evolution in time is described using *n* variables of time nature: $\tau_i = \varepsilon^i \tau, \quad i = 0,1,2,\ldots$, where $\varepsilon$ is a small parameter. The differential operators of the new variables are redefined according to the chain rule and the solution $\varphi(\tau)$ to the initial value problem (2)–(3) is sought in the form of a power series of the small parameter

$$\varphi(\tau;\varepsilon) = \sum_{k=1}^{n} \varepsilon^k \phi_k(\tau_0, \tau_1, \tau_2, \ldots) + O(\varepsilon^{n+1}), \tag{4}$$

where *n* denotes the number of time scales. For the initial value problem (2)–(3), at least $n = 3$ should be assumed. Assuming weak damping in the form

$$\alpha_1 = \varepsilon^2 \hat{\alpha}_1, \;\; \alpha_2 = \varepsilon \hat{\alpha}_2, \tag{5}$$

and omitting the terms that are accompanied by $\varepsilon$ in powers higher than three, one can obtain the set of three differential equations

$$\frac{\partial^2 \phi_1}{\partial \tau_0^2} + \phi_1 = 0, \tag{6}$$

$$\frac{\partial^2 \phi_2}{\partial \tau_0^2} + 2\frac{\partial^2 \phi_1}{\partial \tau_0 \, \partial \tau_1} + \phi_2 = 0, \tag{7}$$

$$\frac{1}{6}\phi_1^3 - \phi_3 = \frac{\partial^2 \phi_1}{\partial \tau_1^2} + \hat{\alpha}_1 \frac{\partial \phi_1}{\partial \tau_0} + \hat{\alpha}_2 \frac{\partial \phi_1}{\partial \tau_0}\left|\frac{\partial \phi_1}{\partial \tau_0}\right| + 2\frac{\partial^2 \phi_1}{\partial \tau_0 \, \partial \tau_2} + 2\frac{\partial^2 \phi_2}{\partial \tau_0 \, \partial \tau_1} + \frac{\partial^2 \phi_3}{\partial \tau_0^2}. \tag{8}$$

## 3. Concluding Remarks

The equations of the first (6) and second (7) order approximation have been solved many times in the literature [3, 4, 5]. In the equation of the third order approximation (8) there is an absolute value term which significantly complicates the solution of the problem. The damping coefficients for this problem were determined in paper [6] .These results will be used to compare the results obtained by the multiple scale method with the experimental results.

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